Eur. Phys. J. C 16, 161–167 (2000)
Digital Object Identifier (DOI) 10.1007/s100520000394

THE EUROPEAN
PHYSICAL JOURNAL C

Gauge theory on noncommutative spaces

J. Madore[1,2], S. Schraml[2,3], P. Schupp[3], J. Wess[2,3]

[1] Laboratoire de Physique Théorique, Université de Paris-Sud, Bâtiment 211, 91405 Orsay, France
[2] Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany
[3] Sektion Physik, Universität München, Theresienstrasse 37, 80333 München, Germany

Received: 6 March 2000 / Published online: 8 May 2000 – © Springer-Verlag 2000

Abstract. We introduce a formulation of gauge theory on noncommutative spaces based on the notion of covariant coordinates. Some important examples are discussed in detail. A Seiberg-Witten map is established in all cases.

1 Introduction

We introduce a natural method to formulate a gauge theory on more or less arbitrary noncommutative spaces. The starting point is the observation that multiplication of a (covariant) field by a coordinate can in general not be a covariant operation in noncommutative geometry, because the coordinates will not commute with the gauge transformations. The idea is to make the coordinates covariant by adding a gauge potential to them. This is analogous to the case in usual gauge theory; one adds gauge potentials to the partial derivatives to obtain covariant derivatives. One can consider a covariant coordinate as a position-space analogue of the usual covariant momentum of gauge theory.

In the following we prefer not to present the general case of an arbitrary associative algebra of noncommuting variables; we consider rather three important examples in which the commutator of two coordinates is respectively constant, linear and quadratic in the coordinates. We employ Weyl's quantization procedure to associate with an algebra of noncommuting coordinates an algebra of functions of commuting variables with deformed product. One of our examples gives the same kind of noncommutative gauge theory that has appeared in string theory recently [1].

2 Covariant coordinates

The associative algebraic structure $\mathcal{A}_x$ which defines a noncommutative space can be defined in terms of a set of generators $\hat{x}^i$ and relations $\mathcal{R}$. Instead of considering a general expression for the relations we shall discuss rather some important explicit cases. These are of the form of a canonical structure

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \qquad \theta^{ij} \in \mathbb{C}, \tag{2.1}$$

a Lie-algebra structure

$$[\hat{x}^i, \hat{x}^j] = iC^{ij}{}_k \hat{x}^k, \qquad C^{ij}{}_k \in \mathbb{C}, \tag{2.2}$$

and a quantum space structure [2–5]

$$\hat{x}^i \hat{x}^j = q^{-1} \hat{R}^{ij}{}_{kl} \hat{x}^k \hat{x}^l, \qquad \hat{R}^{ij}{}_{kl} \in \mathbb{C}. \tag{2.3}$$

In all these cases the index i takes values from 1 to N. We shall suppose that $\mathcal{A}_x$ has a unit element. For the quantum space structure a simple version is the Manin plane, with $N = 2$:

$$\hat{x}\hat{y} = q\hat{y}\hat{x}, \qquad q \in \mathbb{C}. \tag{2.4}$$

We shall refer to the generators $\hat{x}^i$ of the algebra as 'coordinates' and we shall consider $\mathcal{A}_x$ to be the algebra of formal power series in the coordinates modulo the relations

$$\mathcal{A}_x \equiv \mathbb{C}\left[[\hat{x}^1, \ldots, \hat{x}^N]\right]/\mathcal{R}. \tag{2.5}$$

For a physicist this means that one is free to use the relations (2.1), (2.2) or (2.3), (2.4) to reorder the elements of an arbitrary power series.

We consider fields as elements of the algebra $\mathcal{A}_x$:

$$\psi(\hat{x}) = \psi(\hat{x}^1, \ldots, \hat{x}^N) \in \mathcal{A}_x. \tag{2.6}$$

We shall introduce the notion of an infinitesimal gauge transformation $\delta\psi$ of the field ψ and suppose that under an infinitesimal gauge transformation $\alpha(\hat{x})$ it can be written in the form

$$\delta\psi(\hat{x}) = i\alpha(\hat{x})\psi(\hat{x}); \qquad \alpha(\hat{x}),\ \psi(\hat{x}) \in \mathcal{A}_x. \tag{2.7}$$

This we call a covariant transformation law of a field. It follows then of course that $\delta\psi \in \mathcal{A}_x$. Since $\alpha(\hat{x})$ is an element of $\mathcal{A}_x$ it is the equivalent of an abelian gauge transformation. If $\alpha(\hat{x})$ belonged to an algebra $M_n(\mathcal{A}_x)$ of matrices with elements in $\mathcal{A}_x$ then it would be the equivalent of a non-abelian gauge transformation.

An essential requirement is that the coordinates be invariant under the action of a gauge transformation:

$$\delta\hat{x}^i = 0.$$

Multiplication of a field on the left by a coordinate is then not a covariant operation in the noncommutative case. That is

$$\delta(\hat{x}^i\psi) = i\hat{x}^i\alpha(\hat{x})\psi \tag{2.8}$$

and in general the right-hand side is not equal to $i\alpha(\hat{x})\hat{x}^i\psi$. Following the ideas of ordinary gauge theory we introduce covariant coordinates $\hat{X}^i$ such that

$$\delta(\hat{X}^i\psi) = i\alpha\hat{X}^i\psi, \tag{2.9}$$

that is, $\delta(\hat{X}^i) = i[\alpha, \hat{X}^i]$. To find the relation between $\hat{X}^i$ and $\hat{x}^i$ we make an Ansatz of the form

$$\hat{X}^i = \hat{x}^i + A^i(\hat{x}), \qquad A^i(\hat{x}) \in \mathcal{A}_x. \tag{2.10}$$

This is quite analogous to the expression of a covariant derivative as the sum of an ordinary derivative plus a gauge potential.[1]

We derive the transformation properties of A^i from the requirement (2.9):

$$\delta A^i = i[\alpha, A^i] - i[\hat{x}^i, \alpha]. \tag{2.11}$$

The right hand side can be evaluated using one of the relations (2.1), (2.2) or (2.3). It is easy to see that a tensor T^{ij} can be defined in each case as respectively

$$T^{ij} = [\hat{X}^i, \hat{X}^j] - i\theta^{ij} \tag{2.12}$$

in the canonical case,

$$T^{ij} = [\hat{X}^i, \hat{X}^j] - iC^{ij}{}_k\hat{X}^k \tag{2.13}$$

for the Lie-structure and

$$T^{ij} = \hat{X}^i\hat{X}^j - q^{-1}\hat{R}^{ij}{}_{kl}\hat{X}^k\hat{X}^l \tag{2.14}$$

for the quantum space.[2] We verify directly that the objects T^{ij} are covariant tensors. In the canonical case we find

$$\begin{aligned} T^{ij} &= [A^i, \hat{x}^j] + [\hat{x}^i, A^j] + [A^i, A^j], \\ \delta T^{ij} &= [\delta A^i, \hat{x}^j] + [\hat{x}^i, \delta A^j] + [\delta A^i, A^j] + [A^i, \delta A^j]. \end{aligned} \tag{2.15}$$

[1] There is a ‘dual element’ λ_i closely related to the coordinate $\hat{x}^i$ and defined so that the inner derivation $\mathrm{ad}\,\lambda_i$ of $\mathcal{A}_x$ plays the role of the ordinary derivative. In this context a general consistency relation for the λ_i has been given [6,8] which also covers the relations (2.1), (2.2) and (2.3). This relation states that when the $\hat{X}^i$ vanish the T^{ij} must lie in the center of the algebra.

[2] These are not the only covariant objects which can be constructed from the $\hat{X}^i$ but they have a natural geometric significance as gauge field strengths. The second expression (2.13), for example, has a direct interpretation [7] as the field strength of an abelian gauge potential over a geometry with $\mathcal{A}_x = M_n$, the algebra of $n \times n$ matrices.

We insert δA^i from (2.11), use the Jacobi identity and obtain

$$\delta T^{ij} = i[\alpha, T^{ij}]. \tag{2.16}$$

Exactly the same procedure leads to the result for the Lie structure:

$$\begin{aligned} T^{ij} &= [\hat{x}^i, A^j] + [A^i, \hat{x}^j] + [A^i, A^j] - iC^{ij}{}_kA^k, \\ \delta T^{ij} &= i[\alpha, T^{ij}]. \end{aligned} \tag{2.17}$$

In the case of the quantum space we find

$$T^{ij} = P^{ij}{}_{kl}(A^k\hat{x}^l + \hat{x}^kA^l + A^kA^l) \tag{2.18}$$

where we have introduced P defined as

$$P^{ij}{}_{kl} = \delta^i_k\delta^j_l - q^{-1}\hat{R}^{ij}{}_{kl}. \tag{2.19}$$

We again insert δA^i from (2.11) to compute δT^{ij}. We obtain

$$\begin{aligned} \delta T^{ij} = iP^{ij}{}_{kl}\big\{&[\alpha, A^k]\hat{x}^l + [\alpha, \hat{x}^k]\hat{x}^l + \hat{x}^k[\alpha, A^l] + \hat{x}^k[\alpha, \hat{x}^l] \\ &+[\alpha, A^k]A^l + [\alpha, \hat{x}^k]A^l \\ &+A^k[\alpha, A^l] + A^k[\alpha, \hat{x}^l]\big\}. \end{aligned} \tag{2.20}$$

With relation (2.3) this becomes

$$\delta T^{ij} = i[\alpha, T^{ij}]. \tag{2.21}$$

3 Weyl quantization

In the framework of canonical quantization Hermann Weyl [9] gave a prescription how to associate an operator with a classical function of the canonical variables. This prescription can also be used to associate an element of $\mathcal{A}_x$ with a function f of classical variables $x^1, \dots x^n$ [10]. We use $\hat{x}$ for elements of $\mathcal{A}_x$ and x for the associated classical commuting variables. Using the Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int d^nx\, e^{-ik_jx^j}f(x) \tag{3.1}$$

of the function $f(x^1, \dots x^n)$ we define an operator

$$W(f) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int d^nk\, e^{ik_j\hat{x}^j}\tilde{f}(k). \tag{3.2}$$

This is a unique prescription, the operator $\hat{x}$ replaces the variables x in f in the most symmetric way. If the operators $\hat{x}$ have hermiticity properties $W(f)$ will inherit these properties for real f. At present we are interested in the algebraic properties only.

Operators obtained by (3.2) can be multiplied to yield new operators. The question arises whether or not these new operators can be associated also with classical functions. If such a function exists we call it $f \diamond g$ (‘f diamond g’):

$$W(f)W(g) = W(f \diamond g). \tag{3.3}$$

We can write (3.3) more explicitly as

$$W(f)W(g) = \frac{1}{(2\pi)^n} \int d^n k d^n p\, e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} \tilde{f}(k)\tilde{g}(p). \tag{3.4}$$

If the product of the two exponentials can be calculated by the Baker-Campbell-Hausdorff formula to give an exponential of a linear combination of the $\hat{x}^i$ the function $f \diamond g$ will exist. This is the case for the canonical structure:

$$e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} = e^{i(k_j + p_j)\hat{x}^j - \frac{i}{2} k_i p_j \theta^{ij}}. \tag{3.5}$$

A comparison with (3.2) shows that $(f \diamond g)(x)$ can be computed from (3.4) and (3.5) by replacing $\hat{x}$ by x.

$$\begin{aligned} f \diamond g &= \\ f * g &= \frac{1}{(2\pi)^n} \int d^n k d^n p\, e^{i(k_j+p_j)x^j - \frac{i}{2} k_i \theta^{ij} p_j} \tilde{f}(k)\tilde{g}(p) \\ &= e^{\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{ij} \frac{\partial}{\partial y^j}} f(x) g(y) \Big|_{y \to x} \end{aligned} \tag{3.6}$$

We obtain the Moyal-Weyl $*$-product [11].

A similar $*$-product is obtained for the Lie structure:

$$e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} = e^{iP_i(k,p)\hat{x}^i} \tag{3.7}$$

where $P_i(k,p)$ are the parameters of a group element obtained by multiplying two group elements, one parametrized by k and the other by p. From the Baker-Campbell-Hausdorff formula we know that

$$P_i(k,p) = k_i + p_i + \frac{1}{2} g_i(k,p) \tag{3.8}$$

where g_i contains the information about the noncommutative structure of the group. Again we find the star product after a Fourier transformation

$$\begin{aligned} f \diamond g = f * g &= \frac{1}{(2\pi)^n} \int d^n k d^n p\, e^{iP_i(k,p)x^i} \tilde{f}(k)\tilde{g}(p) \\ &= e^{\frac{i}{2} x^i g_i(i \frac{\partial}{\partial y}, i \frac{\partial}{\partial z})} f(y) g(z) \Big|_{\substack{y \to x \\ z \to x}}. \end{aligned} \tag{3.9}$$

A more complicated situation arises for the quantum plane structure. The Baker-Campbell-Hausdorff formula cannot be used explicitly. The Weyl quantization (3.2) does not seem to be the most natural one. At the moment we are only interested in the algebraic structure of the theory. In this context any unique way of associating an operator with a function of the classical variables would do. For the quantum plane this could be a normal ordering. We treat the case of the Manin plane (2.4) explicitly. With any monomial in $x\, y$ we associate the normal ordered product of the operators $\hat{x}$, $\hat{y}$ where all the $\hat{x}$ operators are placed to the left and all the $\hat{y}$ operators to the right:

$$W(f(x,y)) = :f(\hat{x},\hat{y}): \tag{3.10}$$

The dots indicate the above normal ordering. Equation (3.3) now has to be written in the form:

$$:f(\hat{x},\hat{y})::g(\hat{x},\hat{y}): = :f \diamond g(\hat{x},\hat{y}): \tag{3.11}$$

Let us first compute this for monomials:

$$\hat{x}^{n_1} \hat{y}^{m_1} \hat{x}^{n_2} \hat{y}^{m_2} = q^{-m_1 n_2} \hat{x}^{n_1+n_2} \hat{y}^{m_1+m_2} \tag{3.12}$$

$$\begin{aligned} :\hat{x}^{n_1} \hat{y}^{m_1}::\hat{x}^{n_2} \hat{y}^{m_2}: &= q^{-m_1 n_2} :\hat{x}^{n_1+n_2} \hat{y}^{m_1+m_2}: \\ &= W\left(q^{-x' \frac{\partial}{\partial x'} y \frac{\partial}{\partial y}} x^{n_1} y^{m_1} x'^{n_2} y'^{m_2} \Big|_{\substack{x' \to x \\ y' \to y}} \right) \end{aligned}$$

This is easily generalized to arbitrary power series in x and y

$$f \diamond g = q^{-x' \frac{\partial}{\partial x'} y \frac{\partial}{\partial y}} f(x,y) g(x',y') \Big|_{\substack{x' \to x \\ y' \to y}} \tag{3.13}$$

and we have obtained a diamond product for the Manin plane. Instead of the $\hat{x}\hat{y}$ ordering we could have used the $\hat{y}\hat{x}$ ordering or, more reasonably, the totally symmetric product of the $\hat{x}\hat{y}$ operators. For monomials of fixed degree the $\hat{x}\hat{y}$ ordered and the $\hat{y}\hat{x}$ ordered as well as the symmetrically ordered products form a basis. Thus the diamond product exists in all the cases and it is only a combinatorial problem to compute it explicitly.

Weyl quantization allows the representation of an element of $\mathcal{A}_x$ by a classical function of x. For a constant c and for $\hat{x}^i \in \mathcal{A}_x$ this is trivial:

$$c \to c, \qquad \hat{x}^i \to x^i. \tag{3.14}$$

The formula (3.3) can be used to generalize this to any element of $\mathcal{A}_x$. As an example we take the bilinear elements of $\mathcal{A}_x$.

$$\hat{x}^i \hat{x}^j = W(x^i)W(x^j) = W(x^i \diamond x^j), \qquad \hat{x}^i \hat{x}^j \to x^i \diamond x^j. \tag{3.15}$$

In particular

$$W(x^i x^j) = \frac{1}{2}(\hat{x}^i \hat{x}^j + \hat{x}^j \hat{x}^i) \tag{3.16}$$

for the canonical structure and the Lie structure. For the quantum space structure we have

$$W(x^i x^j) = :\hat{x}^i \hat{x}^j: \tag{3.17}$$

The elements of $\mathcal{A}_x$ can be represented by functions $f(x)$, the multiplication of the elements by the star product of the functions. This product is associative. Let us now represent a field by a classical function $\psi(x)$. The gauge transformation (2.7) is represented by $\alpha(x)$:

$$\delta_\alpha \psi(x) = i\alpha(x) \diamond \psi(x). \tag{3.18}$$

We immediately conclude that

$$\begin{aligned} (\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha)\psi(x) &= i\beta(x) \diamond (\alpha(x) \diamond \psi(x)) \\ &\quad - i\alpha(x) \diamond (\beta(x) \diamond \psi(x)) \\ &= i\,(\beta \diamond \alpha - \alpha \diamond \beta) \diamond \psi. \end{aligned} \tag{3.19}$$

The transformation law of $A^i(x)$, representing the element $A^i \in \mathcal{A}_x$ is:

$$\delta A^i = i[\alpha \stackrel{\diamond}{,} A^i] - i[x^i \stackrel{\diamond}{,} \alpha] \tag{3.20}$$

and for the tensors $T^{ij}(x)$:

$$\delta T^{ij} = i[\alpha \,\overset{\diamond}{,}\, T^{ij}]. \tag{3.21}$$

Where T^{ij} is defined as in (2.12), (2.13), (2.14), but with elements of $\mathcal{A}_x$ and algebraic multiplication replaced by the corresponding functions and diamond product:

$$\begin{aligned} T^{ij} &= [A^i \,\overset{\diamond}{,}\, x^j] + [x^i \,\overset{\diamond}{,}\, A^j] + [A^i \,\overset{\diamond}{,}\, A^j] \\ T^{ij} &= [x^i \,\overset{\diamond}{,}\, A^j] + [A^i \,\overset{\diamond}{,}\, x^j] + [A^i \,\overset{\diamond}{,}\, A^j] - iC^{ij}{}_k A^k \\ T^{ij} &= P^{ij}{}_{kl}(A^k \diamond x^l + x^k \diamond A^l + A^k \diamond A^l). \end{aligned} \tag{3.22}$$

4 Noncommutative gauge theories

4.1 Canonical structure

We would now like to give explicit formulae for the gauge transformation and tensor in the canonical case and will explain the relation to the conventions of noncommutative Yang-Mills theory as presented in [1]. The commutator $[\hat{x}^i, .]$ in the transformation of a gauge potential (2.11),

$$\delta A^i = -i[\hat{x}^i, \alpha] + i[\alpha, A^i],$$

acts as a derivation on elements of $\mathcal{A}_x$. Due to the special form of the commutation relations (2.1) with the *constant* θ^{ij}, this commutator can in fact be written as a derivative on elements $f \in \mathcal{A}_x$:

$$[\hat{x}^i, f] = i\theta^{ij}\partial_j f. \tag{4.1}$$

The derivative ∂_j is defined as a derivation on $\mathcal{A}_x$, that is, $\partial_j fg = (\partial_j f)g + f(\partial_j g)$ and on the coordinates as: $\partial_j \hat{x}^i \equiv \delta^i_j$. The right-hand side of (4.1) is a derivation because the matrix θ is constant and commutes with everything. We find that in the canonical case the gauge transformation can be written

$$\delta A^i = \theta^{ij}\partial_j\alpha + i[\alpha, A^i]. \tag{4.2}$$

The gauge potential $\hat{A}$ of noncommutative Yang-Mills is introduced by the identification

$$A^i \equiv \theta^{ij}\hat{A}_j. \tag{4.3}$$

We must here assume that the matrix θ is non-degenerate. We find the following transformation law for the gauge field $\hat{A}_j$:

$$\delta\hat{A}_j = \partial_j\alpha + i[\alpha, \hat{A}_j]. \tag{4.4}$$

It has exactly the same form as the transformation law for a non-abelian gauge potential in commutative geometry, except that in general the meaning of the commutator is different. An explicit expression for the tensor T in the canonical case (2.15) is found likewise,

$$T^{ij} = i\theta^{ik}\partial_k A^j - i\theta^{jl}\partial_l A^i + [A^i, A^j]. \tag{4.5}$$

Up to a factor i, the relation to the field strength $\hat{F}$ of noncommutative Yang-Mills is again simply obtained by using θ to raise indices:

$$T^{ij} \equiv i\theta^{ik}\theta^{jl}\hat{F}_{kl}. \tag{4.6}$$

Assuming again non-degeneracy of θ, we find

$$\hat{F}_{kl} = \partial_k\hat{A}_l - \partial_l\hat{A}_k - i[\hat{A}_k, \hat{A}_l]. \tag{4.7}$$

According to our conventions we are to consider this as the field strength of an abelian gauge potential in a noncommutative geometry, but except for the definition of the bracket it has again the same form as a non-abelian gauge field-strength in commutative geometry. Since $\theta^{ij} \in \mathbb{C}$, $\hat{F}$ is a tensor:

$$\delta\hat{F}_{kl} = i[\alpha, \hat{F}_{kl}]. \tag{4.8}$$

These formulae become clearer and the relation to noncommutative Yang-Mills theory is even more direct, if we represent the elements of $\mathcal{A}_x$ by functions of the classical variables x^i and use the Moyal-Weyl star product (3.6). In particular equation (4.1) becomes

$$x^i * f - f * x^i = i\theta^{ij}\partial_j f, \tag{4.9}$$

where $f(x)$ is now a function and $\partial_j f = \partial f/\partial x^j$ is the ordinary derivative. This follows directly from the Moyal-Weyl product (3.6). The identifications (4.3,4.6) have the same form as before. The relevant equations written in terms of the star product become

$$\delta A^i = \theta^{ij}\partial_j\alpha + i\alpha * A^i - iA^i * \alpha, \tag{4.10}$$

$$T^{ij} = i\theta^{ik}\partial_k A^j - i\theta^{jl}\partial_l A^i + A^i * A^j - A^j * A^i, \tag{4.11}$$

$$\delta T^{ij} = i\alpha * T^{ij} - iT^{ij} * \alpha, \tag{4.12}$$

$$\delta\hat{A}_j = \partial_j\alpha + i\alpha * \hat{A}_j - i\hat{A}_j * \alpha, \tag{4.13}$$

$$\hat{F}_{kl} = \partial_k\hat{A}_l - \partial_l\hat{A}_k - i\hat{A}_k * \hat{A}_l + i\hat{A}_l * \hat{A}_k, \tag{4.14}$$

$$\delta\hat{F}_{kl} = i\alpha * \hat{F}_{kl} - i\hat{F}_{kl} * \alpha \tag{4.15}$$

and

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{(\beta*\alpha - \alpha*\beta)}. \tag{4.16}$$

All this clearly generalizes to A^i, α, $\hat{A}_j$ and $\hat{F}_{kl}$ that are (hermitian) $n \times n$ matrices. We shall return to this point later. It is interesting to note the form of the covariant coordinates written in terms of $\hat{A}$:

$$\hat{X}^i = \hat{x}^i + \theta^{ij}\hat{A}_j. \tag{4.17}$$

This expression has appeared in string theory contexts related to noncommutative Yang-Mills theory mainly as a coordinate transformation [12–14].

Remark. Ordinary gauge theory can be understood as a special case of gauge theory on the noncommutative canonical structure as follows: Consider coordinates $\{\hat{q}^j, \hat{p}_i\}$ with canonical commutation relations $[\hat{q}^j, \hat{p}_i] =$

$i\delta_i^j$ and restrict the allowed choices of infinitesimal gauge transformations α to depend only on the $\hat{q}^i$, i.e., only on half the original coordinates. Multiplying a field ψ by a coordinate is now a non-covariant concept only for half the coordinates, namely for the 'momenta' $\hat{p}_i$. The gauge field A will thus depend only on the $\hat{q}^i$, as will the tensor T. It is not hard to see that the relations of noncommutative gauge theory reduce in this case to those of ordinary gauge theory. The algebra of the $\hat{q}^j$ and $\hat{p}_i$ can of course be realized as ordinary commutative coordinates q^j and derivatives $-i\partial_i$.

4.2 Lie structure

The relations of noncommutative gauge theory on a Lie structure (2.2) written in the language of star products are

$$\delta A^i = -i[x^i \overset{*}{,} \alpha] + i[\alpha \overset{*}{,} A^i], \tag{4.18}$$

$$T^{ij} = [x^i \overset{*}{,} A^j] + [A^i \overset{*}{,} x^j] + [A^i \overset{*}{,} A^j] - iC^{ij}{}_k A^k, \tag{4.19}$$

$$\delta T^{ij} = i\alpha * T^{ij} - iT^{ij} * \alpha, \tag{4.20}$$

where A^i and α are functions of the (commutative) coordinates x^i and the $*$-product is given in (3.9). As in the canonical case, $[x^i \overset{*}{,} f(x)]$ can be written in terms of a derivative of f

$$[x^i \overset{*}{,} f(x)] = iC^{ij}{}_k x^k \frac{\partial f}{\partial x^j}, \tag{4.21}$$

but the proof is not so obvious, because the left-hand side is a derivation of the noncommutative $*$-product while the right-hand side is a derivation with respect to the commutative point-wise product of functions. However, these two notions can be reconciled thanks to the symmetrization inherent in the Weyl quantization procedure. Equations (4.18) and (4.19) can thus also be written as

$$\delta A^i = C^{ij}{}_k x^k \partial_j \alpha + i\alpha * A^i - iA^i * \alpha, \tag{4.22}$$

$$T^{ij} = iC^{il}{}_k x^k \partial_l A^j - iC^{jl}{}_k x^k \partial_l A^i + [A^i \overset{*}{,} A^j] - iC^{ij}{}_k A^k. \tag{4.23}$$

5 Non-abelian gauge transformations

In this case the parameter $\alpha(\hat{x})$ in (2.7) and the gauge field A in (2.10) will be matrix valued:[3] $\alpha = \alpha_r T^r$ and $A = A_r T^r$, where $\alpha_r, A_r \in \mathcal{A}_x$ and the T^r form a suitable basis of matrices. It is not clear what conditions we can consistently impose on these matrices and in particular in which sense they can be Lie-algebra valued; we can, however, always assume that α and A are in the enveloping algebra of a Lie algebra. Let us consider the commutator (2.11). It can be written as a sum of commutators and anti-commutators of the matrices T^i:

$$[\alpha, A] = \frac{1}{2}(\alpha_r A_s + A_s \alpha_r)[T^r, T^s] + \frac{1}{2}(\alpha_r A_s - A_s \alpha_r)\{T^r, T^s\}. \tag{5.1}$$

In the commutative case the second term is zero and it is clear that one can choose T_r from any matrix representation of a Lie algebra. Here, however, α_r and A_s do not commute. As we shall see it is nevertheless possible to consistently impose hermiticity, while it is for example not consistent to impose tracelessness.

Let us now assume that the relations (2.1), (2.2), (2.3) or (2.4) admit a conjugation:

$$(\hat{x}^i)^* = \hat{x}^i \tag{5.2}$$

This will be the case for real θ^{ij}, real $C^{ij}{}_k$ and, in (2.4), q a root of unity. Then it makes sense to speak about "real" functions

$$f^*(\hat{x}) = f(\hat{x}), \tag{5.3}$$

and in this case α could be hermitian:

$$\alpha(\hat{x}) = \alpha_l(\hat{x})T^l = \alpha^*(\hat{x}), \qquad (\alpha_l(\hat{x}))^* = \alpha_l(\hat{x}), \qquad T_l^\dagger = T_l. \tag{5.4}$$

The commutation of those hermitian objects will be anti-hermitian:

$$([\alpha(x), \beta(y)])^* = -[\alpha(x), \beta(y)]. \tag{5.5}$$

We conclude that with α, A and $\hat{x}$ hermitian, δA in (2.11) will be hermitian again. If the matrices T_l form a basis for all hermitian matrices of a certain dimension, then the commutators and anti-commutators in (5.1) will also close into these matrices.

6 Seiberg-Witten map

Seiberg and Witten were able to establish a connection of noncommutative Yang-Mills theory to ordinary Yang-Mills theory. We show that this can be done for all three examples we have considered. We shall consider the more general non-abelian case. The ordinary gauge potential we shall call a_i and the infinitesimal gauge parameter ε. The transformation law of the gauge potential a_i is

$$\delta_\varepsilon a_i = \partial_i \varepsilon + i[\varepsilon, a_i]. \tag{6.1}$$

This has to be compared with the gauge transformation (3.20)

$$\delta A^i = i[\alpha \overset{\diamond}{,} A^i] - i[x^i \overset{\diamond}{,} \alpha]. \tag{6.2}$$

The diamond product can be written in a formal way analogous to deformation quantization [15,16]

$$f \diamond g = fg + \sum_{n \geq 1} h^n B_n(f, g), \tag{6.3}$$

[3] For notational simplicity we are suppressing the index i on A^i.

where the B_n are differential operators bilinear in f and g, and h is an expansion parameter. We are interested in three special cases:
the canonical case

$$f \diamond g = fg + \sum_{n\geq 1} \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{i_1 j_1} \cdots \theta^{i_n j_n} (\partial_{i_1} \cdots \partial_{i_n} f) \times (\partial_{j_1} \cdots \partial_{j_n} g), \tag{6.4}$$

the Lie case

$$\begin{aligned} f \diamond g &= fg + \sum_{n\geq 1} \frac{1}{n!}\left(\frac{i}{2}\sum_k x^k g_k(i\partial_y, i\partial_z)\right)^n f(y)g(z)\bigg|_{\substack{y=x\\z=x}} \\ &= fg + \frac{i}{2} x^k C^{ij}{}_k \partial_i f \partial_j g + \ldots, \end{aligned} \tag{6.5}$$

and the quantum space case (with $h = \ln q$)

$$f \diamond g = fg + \sum_{n\geq 1} \frac{1}{n!}(-h)^n \Big((y\partial_y)^n f \Big)\Big((x\partial_x)^n g \Big). \tag{6.6}$$

The identification with formula (6.3) is obvious. In the following we shall work to second order in h only. For the canonical and the Lie structure the formula for the $*$ commutator is

$$[f \stackrel{*}{,} g] = i\theta^{ij}(x)\partial_i f \partial_j g + \mathcal{O}(\theta^3). \tag{6.7}$$

This expression does not contain any terms in second order in θ. This is typical for a deformation quanization of a Poisson structure [16]. As a consequence the second term on the right-hand side of (6.2) will be:

$$[x^i \stackrel{*}{,} \alpha] = i\theta^{ij}\partial_j \alpha + \mathcal{O}(\theta^3). \tag{6.8}$$

For the canonical and the Lie structure there are in fact no terms of higher than linear order in θ, see (4.9) and (4.21). Following Seiberg and Witten we construct explicitly local expressions A and α in terms of a, ε and θ. This we do by the following Ansatz:

$$\begin{aligned} A^i &= \theta^{ij} a_j + G^i(\theta, a, \partial a, \ldots) + \mathcal{O}(\theta^3), \\ \alpha &= \varepsilon + \gamma(\theta, \varepsilon, \partial\varepsilon, \ldots, a, \partial a, \ldots) + \mathcal{O}(\theta^2), \end{aligned} \tag{6.9}$$

where G^i and γ are of next to leading order in θ. We require that the variation δA^i of (6.9) with the infinitesimal parameter α be obtained from the variation (6.1) of a_i. This is true to first order in θ due to the Ansatz (6.9). In second order we obtain an equation for G^i and γ:

$$\begin{aligned} \delta_\varepsilon G^i = \theta^{ij}\partial_j \gamma - \tfrac{1}{2}\theta^{kl}\left\{\partial_k \varepsilon, \partial_l(\theta^{ij}a_j)\right\} \\ + i[\varepsilon, G^i] + i[\gamma, \theta^{ij}a_j]. \end{aligned} \tag{6.10}$$

This equation has the following solution:

$$\begin{aligned} G^i &= -\tfrac{1}{4}\theta^{kl}\{a_k, \partial_l(\theta^{ij}a_j) + \theta^{ij}F_{lj}\}, \\ \gamma &= \tfrac{1}{4}\theta^{lm}\{\partial_l \varepsilon, a_m\}, \end{aligned} \tag{6.11}$$

where F_{ij} is the classical field strength $F_{ij} = \partial_i a_j - \partial_j a_i - i[a_i, a_j]$. To prove that this indeed solves Equation (6.10), one has to use the Jacobi identity for $\theta^{ij}(x)$. In the canonical case, that is with θ^{ij} constant, this is the same result as found previously [1] if one takes into account the identification (4.3).

Our quantum space example does not fit into the framework of deformation quantization as specified by (6.7); a quadratic term in $h = \ln q$ appears:

$$\begin{aligned} [f \stackrel{\diamond}{,} g] = &-hxy(\partial_y f \partial_x g - \partial_y g \partial_x f) \\ &+\frac{1}{2}h^2 xy\left\{(\partial_y f \partial_x g - \partial_y g \partial_x f)\right. \\ &+xy(\partial_y^2 f \partial_x^2 g - \partial_y^2 g \partial_x^2 f) \\ &+x(\partial_y f \partial_x^2 g - \partial_y g \partial_x^2 f) \\ &\left.+y(\partial_y^2 f \partial_x g - \partial_y^2 g \partial_x f)\right\} \end{aligned} \tag{6.12}$$

This has as a consequence that a second order term will appear in the following formula:

$$\begin{aligned} [x \stackrel{\diamond}{,} \alpha] &= +hxy\partial_y \alpha - \tfrac{1}{2}h^2 xy\partial_y(y\partial_y \alpha) \\ [y \stackrel{\diamond}{,} \alpha] &= -hxy\partial_x \alpha + \tfrac{1}{2}h^2 xy\partial_x(x\partial_x \alpha). \end{aligned} \tag{6.13}$$

Nevertheless a Seiberg-Witten map can be constructed – at least for the abelian case. The transformation is

$$\begin{aligned} A^x &= -ihxya^y + \frac{1}{2}h^2 xy\left[\partial_y(xa^x(ya^y - i))\right. \\ &\quad \left.-\partial_x(xya^y a^y)\right] + \mathcal{O}(h^3) \\ A^y &= +ihxya^x + \frac{1}{2}h^2 xy\left[\partial_x(ya^y(xa^x + i))\right. \\ &\quad \left.-\partial_y(xya^x a^x)\right] + \mathcal{O}(h^3) \\ \alpha &= \varepsilon + \frac{1}{2}h\left[y\partial_y \alpha + x\partial_x \alpha + ixy(a_x \partial_y \alpha - a_y \partial_x \alpha)\right] \\ &\quad +\mathcal{O}(h^2). \end{aligned} \tag{6.14}$$

This suggests that there should be an underlying geometric interpretation of the Seiberg-Witten map.

References

1. N. Seiberg, E. Witten, String Theory and Noncommutative Geometry, JHEP **9909**, 032 (1999), hep-th/9908142
2. J. Wess, q-deformed Heisenberg Algebras, math-ph/99100013
3. B. L. Cerchiai, R. Hinterding, J. Madore, J. Wess, A calculus based on a q-deformed Heisenberg Algebra, Eur. Phys. J. C **8**, 547 (1999), math/9809160
4. J. Wess, B. Zumino, Covariant Differential Calculus on the Quantum Hyperplane, Nucl. Phys.B (Proc. Suppl) **18 B**, 302 (1990)
5. W. B. Schmidke, J. Wess, B. Zumino, A q-deformed Lorentz Algebra, Z. Phys. C **52**, 471 (1991)
6. A. Dimakis, J. Madore, Differential calculi and linear connections, J. Math. Phys. **37**, 4647 (1996)
7. M. Dubois-Violette, R. Kerner, J. Madore, Gauge bosons in a noncommutative geometry, Phys. Lett. B **217**, 485 (1989)

8. J. Madore, An introduction to noncommutative differential geometry and its physical applications. No. 257 in London Mathematical Society Lecture Note Series. Cambridge University Press, second ed., 1999
9. H. Weyl, Quantenmechanik und Gruppentheorie, Z. Physik **46**, 1 (1927); The theory of groups and quantum mechanics, Dover, New-York (1931), translated from Gruppentheorie und Quantenmechanik, Hirzel Verlag, Leipzig (1928)
10. E. P. Wigner, Quantum corrections for thermodynamic equilibrium, Phys. Rev. **40**, 749 (1932)
11. J. E. Moyal, Quantum mechanics as a statistical theory, Proc. Cambridge Phil. Soc. **45**, 99 (1949)
12. L. Cornalba, D-brane Physics and Noncommutative Yang-Mills Theory, hep-th/9909081
13. N. Ishibashi, A Relation between Commutative and Noncommutative Descriptions of D-branes, hep-th/9909176
14. B. Jurčo, P. Schupp, Noncommutative Yang-Mills from equivalence of star products, hep-th/0001032
15. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, Deformation theory and quantization. I. Deformations of symplectic structures, Ann. Physics **111**, 61 (1978)
16. M. Kontsevich, Deformation quantization of Poisson manifolds, I, hep-th/9707040